University of California, Berkeley

Physics 110A Spring 2003 Section 2 (*Strovink*)

Solution Set 1

1. (a) Since $(AB)_{ij} = A_{ik}B_{kj}$ and $(A^t)_{ij} = A_{ji}$, we have that

$$((AB)^t)_{ij} = A_{jk}B_{ki} = B_{ki}A_{jk} = (B^t)_{ik}(A^t)_{kj} = (B^tA^t)_{ij}, \tag{1}$$

which is the identity we wished to prove.

(b) Dropping the summation signs as using the fact that $(R^t)_{ij} = R_{ji}$, Griffiths 1.32 can be written as

$$\overline{T}_{ij} = R_{ik}R_{jl}T_{kl} = R_{ik}T_{kl}R_{jl} = R_{ik}T_{kl}(R^t)_{lj} = (RTR^t)_{ij}. \tag{2}$$

2. We wish to prove that

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \tag{3}$$

To do this directly would entail checking all $3 \times 3 \times 3 \times 3 = 81$ cases, but we can use symmetry and the properties of ϵ_{ijk} to reduce this substantially. First, by rotational symmetry, we can always assume that $i = 1$, which reduces the problem to just checking the remaining 27 possibilities for j, k, and l. Next, note that both sides of the equation are antisymmetric under the interchange of i and j. This means that if $i = j = 1$, both sides are zero, leaving us only to check the $2 \times 3 \times 3 = 18$ cases with $i = 1$ and $j = 2, 3$,

$$\epsilon_{1jk}\epsilon_{klm} = \delta_{1l}\delta_{jm} - \delta_{1m}\delta_{jl}. \tag{4}$$

Similarly, both sides are antisymmetric under the interchange of l and m, and so we only need to check the $2 \times 3 \times 3 - 2 \times 6 = 6$ cases where $m > l$ (so in particular, $m = 2, 3$). In these cases, the right hand side of Eqn. 4 is only non-zero if $l = i = 1$ and also $j = m = 2, 3$. This is also true of the left hand side. To see this, first note that $\epsilon_{1jk} \neq 0$ only if either $j = 2, k = 3$ or $j = 3, k = 2$. In these two cases, $\epsilon_{klm} \neq 0$ as well only if $m = j$. Thus, we are left to checking the two remaining non-zero possibilities, $j = m = 2$ and $j = m = 3$,

$$\epsilon_{12k}\epsilon_{k12} \quad = \delta_{11}\delta_{22} - \delta_{12}\delta_{21} \quad = 1, \tag{5}$$

$$\epsilon_{13k}\epsilon_{k13} \quad = \delta_{11}\delta_{33} - \delta_{13}\delta_{31} \quad = 1, \tag{6}$$

which finishes the proof.

3. Using the above identity (3), we see that

$$\begin{aligned}\left[\vec{A} \times (\vec{B} \times \vec{C})\right]_i &= \epsilon_{ijk}\epsilon_{klm}A_jB_lC_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m = A_jB_iC_j - A_jB_jC_i \\ &= B_i(\vec{A} \cdot \vec{C}) - C_i(\vec{A} \cdot \vec{B}) = \left[\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})\right]_i .\end{aligned} \tag{7}$$

4. We use equation (7) with $\vec{A} = \vec{C} = \hat{\mathbf{n}}$ and $\vec{B} = F$,

$$\hat{\mathbf{n}} \times (\vec{F} \times \hat{\mathbf{n}}) = \vec{F}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \vec{F}) = \vec{F} - \hat{\mathbf{n}}(\vec{F} \cdot \hat{\mathbf{n}}). \tag{8}$$

This implies that,

$$\vec{F} = \hat{\mathbf{n}}(\vec{F} \cdot \hat{\mathbf{n}}) + \hat{\mathbf{n}} \times (\vec{F} \times \hat{\mathbf{n}}). \tag{9}$$

Now, the first term on the RHS of this equation is just the component of $\vec{F}$ parallel to $\hat{\mathbf{n}}$, while the second term (which is orthogonal to the first as it $\hat{\mathbf{n}}$ times another vector) is just the component perpendicular to it.

5. (a) If $S_{ij} = S_{ji}$ and $A_{ij} = -A_{ji}$, then,

$$S_{ij}A_{ij} = S_{ji}(-A_{ji}). \tag{10}$$

Now, as we sum over the indices i and j (these are referred to as dummy indices), what we choose to call them doesn't really matter. For instance, I could replace j with k and i with l, and clearly, $S_{ji}(-A_{ji}) = S_{kl}(-A_{kl})$. So now, nothing stops me from further replacing k with i and l with j, to get $S_{kl}(-A_{kl}) = S_{ij}(-A_{ij}) = -S_{ij}A_{ij}$. But now, tracing back to the first equation, we get

$$S_{ij}A_{ij} = -S_{ij}A_{ij} = 0. \tag{11}$$

(b) Since ϵ_{ijk} is antisymmetric in j and k while $\partial_j\partial_k f$ is symmetric in j and k, the result of part (a) immediately gives us,

$$\left[(\nabla \times (\nabla f))\right]_i = \epsilon_{ijk}\partial_j\partial_k f = 0. \tag{12}$$

(c) Since ϵ_{ijk} is antisymmetric in i and j while $\partial_i\partial_j F_k$ is symmetric in i and j, again, from part (a) we have,

$$\nabla \cdot (\nabla \times \vec{F}) = \epsilon_{ijk}\partial_i\partial_j F_k = 0. \tag{13}$$

6. We consider the vector field $\vec{F}(\vec{r}) = \hat{\boldsymbol{\phi}}$.

(a) Since C is a circle of radius r in the xy plane, the line element along C is a vector tangent to the circle (so along $\hat{\boldsymbol{\phi}}$) with magnitude given by an infinitessimal element of circumference. In particular, using Griffiths 1.68 with $\theta = \pi/2$, we see that $d\vec{l} = (rd\phi)\hat{\boldsymbol{\phi}}$. Thus, we have that

$$\oint_C \vec{F} \cdot d\vec{l} = \int_0^{2\pi} \hat{\boldsymbol{\phi}} \cdot (rd\phi)\hat{\boldsymbol{\phi}} = \int_0^{2\pi} rd\phi = 2\pi r. \tag{14}$$

(b) First, we calculate the curl of $\vec{F}$ in spherical coordinates using Griffiths 1.72 with $F_\phi = 1$ and $F_\theta = F_r = 0$,

$$\nabla \times \vec{F} = \frac{\cos\theta}{r\sin\theta}\hat{\mathbf{r}} - \frac{1}{r}\hat{\boldsymbol{\theta}}. \tag{15}$$

Next, we need the area element for integrating over the surface of a (hemi-)sphere. Since r is constant while θ and ϕ vary, we see that (from Griffiths page 40),

$$d\vec{a} = dl_\theta dl_\phi \hat{\mathbf{r}} = r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}. \tag{16}$$

Integrating over the hemisphere means that we restrict the integral over θ to $(0, \pi/2)$ rather than $(0, \pi)$, so we find that

$$\int_H (\nabla \times \vec{F}) \cdot d\vec{a} = \int_0^{\pi/2} \frac{\cos\theta}{r\sin\theta} r^2 \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi r \int_0^{\pi/2} \cos\theta = 2\pi r. \tag{17}$$

(c) Now, instead of integrating over the hemisphere, we need to integrate over a disk in the xy plane. Here $\theta = \pi/2$ is constant, while we vary ϕ and r, and we expect that the area element should be directed along $\hat{\boldsymbol{\theta}}$. However, $\hat{\boldsymbol{\theta}}$ increases as we got to negative z, so it points downward on the xy-plane while we want an upward pointing area element. Thus, we are lead to the area element (- the result from Griffiths page 40),

$$d\vec{a} = -dl_r dl_\phi \hat{\boldsymbol{\theta}} = -rdrd\phi\hat{\boldsymbol{\theta}}. \tag{18}$$

Thus, we find that

$$\int_D (\nabla \times \vec{F}) \cdot d\vec{a} = \int_0^r \frac{-1}{r}(-r)dr \int_0^{2\pi} d\phi = 2\pi r. \tag{19}$$

(d) Clearly, as the answer from part (a), (b), and (c) all agree, and as C is the boundary for both the Hemisphere and Disk (H and D), we see that Stoke's theorem indeed holds.

7. We use Dirac delta functions to express various charge distributions as three dimensional charge densities.

(a) A charge Q distributed over a disk of radius b in the surface $x = y = 0$ should have a charge density proportional to $Q/(\pi b^2)$ and be non-zero only if $z = 0$ and $r < b$ in cylindrical coordinates. Thus, its charge density should be,

$$\rho(\vec{r}) = \frac{Q}{\pi b^2}\theta(b-r)\delta(z), \tag{20}$$

where $\theta(x)$ is a step function (defined in Griffiths 1.95) which is zero if $x \leq 0$ and 1 if $x > 0$. We can check this by integrating this density over all space to make sure that the total charge is actually just Q,

$$\int \rho(\vec{r})d^3\vec{r} = \int_0^\infty \frac{Q}{\pi b^2}\theta(b-r)rdr \int \delta(z)dz \int_0^{2\pi} d\phi = \frac{2Q}{b^2}\int_0^b rdr = Q. \tag{21}$$

(b) An infinitely long wire along the z-axis with charge per unit length λ should correspond to a three dimensional charge density which is non-zero only when $x = y = 0$ and is proportional to λ,

$$\rho(\vec{r}) = \lambda\delta(x)\delta(y). \tag{22}$$

To see that this is the correct expression, we integrate the charge density over a surface $z = a$ for some constant a to get the charge per unit length,

$$\int_{z=a} \rho(\vec{r})dxdy = \int \lambda\delta(x)\delta(y)dxdy = \lambda. \tag{23}$$

(c) A charge per unit length λ distributed over an infinitely long cylinder with radius b along the z-axis should be non-zero only when $r = b$ and all values of ϕ and z in cylindrical coordinates,

$$\rho(\vec{r}) = \frac{\lambda}{2\pi b}\delta(r-b). \tag{24}$$

Just as in part (b), we can integrate this charge density over a surface $z = a$ to get the charge per unit length,

$$\int_{z=a} \rho(\vec{r})rdrd\phi = \int_0^\infty \frac{\lambda}{2\pi b}\delta(r-b)rdr \int_0^{2\pi} d\phi = \lambda. \tag{25}$$

8. We consider the vector field $\vec{H}(x, y, z) = x^2y\hat{\mathbf{x}} + y^2z\hat{\mathbf{y}} + z^2x\hat{\mathbf{z}}$. Since we know that the irrotational part $\vec{F}$ obeys $\nabla \cdot \vec{H} = \nabla \cdot \vec{F}$, and that $\vec{F} = -\nabla V$ for some potential V, we must have

$$\nabla \cdot \nabla V = \nabla^2 V = -\nabla \cdot \vec{H} = -2(xy + yz + zx). \tag{26}$$

So, we need to find some $V(x, y, z)$ which satisfies

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -2(xy + yz + zx) \tag{27}$$

Clearly, the following guess does the job,

$$V(x, y, z) = -\left[x^2(yz) + y^2(xz) + z^2(xy)\right]. \tag{28}$$

Thus, we find that

$$\vec{F} = -\nabla V = yz(2x + y + z)\hat{\mathbf{x}} + xz(2y + x + z)\hat{\mathbf{y}} + xy(2z + x + y)\hat{\mathbf{z}}, \tag{29}$$

and so we must have

$$\vec{G} = \vec{H} - \vec{F} = y(x^2 - z(2x + y + z))\hat{\mathbf{x}} + z(y^2 - x(2y + x + z))\hat{\mathbf{y}} + x(z^2 - (2z + x + y))\hat{\mathbf{z}}, \tag{30}$$

where by construction, $\nabla \cdot \vec{G} = 0$ and so $\vec{G}$ is solenoidal.